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The symmetric equilibria of symmetric voter participation games with complete information

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ABSTRACT

We characterize the symmetric Nash equilibria of the symmetric voter participation game with complete information from Palfrey and Rosenthal (1983). To do so, we use methods based on polynomials in Bernstein form to determine how the probability that a voter is pivotal depends on the participation probability and the number of players in the game.

1. Introduction

In a seminal contribution to the theory of voter turnout,[1] Palfrey and Rosenthal (1983) introduce a participation game in which each player decides whether to cast a vote for her favorite among two candidates or to abstain. Participation costs $0 < c < 1/2$ and the benefit of the favored candidate winning the election, with the latter normalized to one, are the same for all players. Further, there is complete information about the numbers of players favoring each of the two candidates. Palfrey and Rosenthal (1983) show that this model typically has many Nash equilibria, including some featuring substantial turnout. These observations have proved highly influential, triggering a substantial game-theoretic literature exploring variations of the voter participation game, including Palfrey and Rosenthal (1985), Myerson (1998), and Börgers (2004), who introduce various sources of incomplete information.

Despite the prominence and importance of the voter participation game considered in Palfrey and Rosenthal (1983), the formal analysis of their model has remained incomplete. In particular, this is the case for the characterization of the totally mixed symmetric equilibria (in which all players vote with the same probability $0 < x < 1$) of the symmetric voter participation game (in which both candidates have the same number of supporters $s \geq 2$ and ties are broken with a fair coin toss) that we will consider in this note.

The totally mixed symmetric equilibria of the symmetric voter participation game are discussed in Section 6.A of Palfrey and Rosenthal (1983). They characterize the participation cost, which we will denote by $\underline{c}(s)$, with the property that there is an equilibrium in which all players vote with probability $x = 1/2$ and show that $\underline{c}(s)$ converges to zero as s goes to infinity

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[1] See Dhillon and Peralta (2002) and Feddersen (2004) for surveys.

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(Palfrey and Rosenthal, 1983, Proposition 6). In addition, they offer numerical calculations (illustrated by their Figs. 3 and 4) and a conjecture (Palfrey and Rosenthal, 1983, p. 29) suggesting the following:

1. For $c < \underline{c}(s)$ there is no totally mixed symmetric equilibrium.
2. For $c = \underline{c}(s)$ there is a unique such equilibrium in which all players vote with probability $x = 1/2$.
3. For $c > \underline{c}(s)$ there are exactly two totally mixed symmetric equilibria, namely a low turnout equilibrium and a high turnout equilibrium with voting probabilities x_{L} and x_{H} satisfying $x_{\mathrm{L}} < 1/2$ and $x_{\mathrm{H}} = 1 - x_{\mathrm{L}} > 1/2$.[2]
4. The voting probability in the low turnout equilibrium converges to 0 and the voting probability in the high turnout equilibrium converges to 1 as the number of players grows large.

In the light of numerical calculations for small s and a heuristic approximation argument for large s (cf. Demichelis and Dhillon, 2010, p. 877–879) these assertions are highly plausible. Though never proven, they have thus been accepted as true by the subsequent literature, most notably in Palfrey and Rosenthal (1985, pp. 65–67), where the symmetric voter participation game serves as the leading example to illustrate the effect of introducing incomplete information about participation costs.

We provide the missing analytical proof for the above assertions and show in addition:

5. The voting probabilities in the low turnout and the high turnout equilibrium change monotonically in the size of the electorate.

The key idea underlying our analysis is to express the pivot probability (i.e., the probability that a player's decision to participate affects the outcome of the election) by first determining the probability that a player is pivotal conditional on the number of other participants in the electorate and then, second, taking expectations over the number of other participants. The resulting expression (Lemma 1) is a polynomial in Bernstein form (cf. Farouki, 2012) with coefficients given by those terms of a hypergeometric probability distribution that correspond to the conditional pivot probabilities. Using well-known properties of the hypergeometric probability distribution we can characterize how the conditional pivot probabilities depend on the number of other participants (Lemma 2) and the size of the electorate (Lemma 3).

Due to their importance in the field of computer-aided design, the properties of polynomials in Bernstein form have been carefully studied and are well understood. It is thus a relatively straightforward task to leverage our results about conditional pivot probabilities to determine the shape of the pivot probability when viewed as a function of the symmetric voting probability x (Proposition 1), to establish that the pivot probability is strictly decreasing in the size of the electorate (Proposition 2), and to show that it converges to zero as s goes to infinity (Proposition 3). Because totally mixed symmetric Nash equilibria are determined by the indifference condition that the expected gain in benefit from voting (which is 1/2 of the pivot probability) is equal to the participation cost c, the above claims 1–5 about the existence, number, location, and comparative statics of such equilibria, follow from these propositions. We formally state these results in Corollaries 1 and 2.

We believe that our contribution goes beyond closing a gap in one of the seminal papers on elections with rational voters. First, recasting the pivot probability in terms of conditional pivot probabilities is not only essential to our approach but is likely to be useful in other voting models. Indeed, conditional pivot probabilities play an important role in Börgers (2004), who considers a model of costly voting in which there is incomplete information about both participation costs and the number of supporters of each alternative.[3] Second, even though economists and game theorists have not taken much note and advantage of this fact, polynomials in Bernstein form arise in many settings other than the one we consider here: in every symmetric binary-action n-player game, in which the payoff to a player depends only on her own choice and the number of other players choosing a particular action, totally mixed symmetric Nash equilibria correspond to the roots of a polynomial in Bernstein form and can thus be investigated with the tools developed for the study of such polynomials. For instance, in Peña et al. (2014) we use the shape-preservation properties of polynomials in Bernstein form to determine the number and evolutionary stability of symmetric equilibria in a broad class of participation games.[4] We are confident that further applications to the analysis of other symmetric, two-action games with many players can be developed.

In the following section we briefly describe the symmetric version of the voter participation game from Palfrey and Rosenthal (1983). Section 3 presents the results. We conclude with a brief discussion. All proofs are in Appendix A.

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[2] The conjecture formulated in Palfrey and Rosenthal (1983) does not exclude the possibility that for some s and $c > \underline{c}(s)$ there might be an even number of totally mixed symmetric equilibria different from two. Their figures, however, strongly suggest that this cannot happen. Palfrey and Rosenthal (1985, p. 66) assert the stronger version of the conjecture as stated here.

[3] Börgers (2004) uses conditional pivot probabilities to show that in his setting the pivot probability is a strictly decreasing function of the ex-ante probability that individuals participate in the vote, cf. the proof of Remark 1 in his paper. Note that, due to the differences in the underlying model, the pivot probabilities in Börgers (2004) are different from the ones appearing in our paper, with our conditional pivot probabilities being u-shaped while his are decreasing.

[4] In Peña et al. (2015) we extend these insights to the biologically relevant cases of interactions in spatially structured populations or between relatives, whereas in Peña and Nöldeke (2016) we use properties of polynomials in Bernstein form to investigate the effects of group-size uncertainty.

2. Model

There are $2s$ voters divided into two teams of equal size $s \geq 2$. Let $T_1 = \{1, \ldots, s\}$ be the members of the first team and $T_2 = \{s+1, \ldots, 2s\}$ be the members of the second team. There are two alternatives (candidates, policy proposals) denoted by A_1 and A_2. Members of team T_1 prefer alternative A_1 and members of T_2 prefer alternative A_2. All players $i = 1, \ldots, 2s$ simultaneously decide whether to vote in favor of their preferred alternative ($a_i = 1$) or to abstain ($a_i = 0$). The election is decided by simple majority rule with ties being broken by a fair coin toss. Players receive a benefit equal to 1 if their favored alternative wins and 0 otherwise. Voting entails a participation cost of $c \in (0, 1/2)$. Players are risk neutral. Hence, for any player i if k other members of her own team and ℓ members of the other team vote and she chooses action $a_i \in \{0, 1\}$, her payoff is

$$\pi(k, \ell, a_i) = \begin{cases} 1 - a_i \cdot c & \text{if } k + a_i > \ell \\ 1/2 - a_i \cdot c & \text{if } k + a_i = \ell \\ -a_i \cdot c & \text{if } k + a_i < \ell. \end{cases}$$

We focus on symmetric mixed strategy profiles in which all voters participate with probability $x \in [0, 1]$. The assumption $c < 1/2$ ensures that the strategy profile in which all agents participate ($x = 1$) is a symmetric Nash equilibrium, whereas the strategy profile in which all agents abstain ($x = 0$) is not.[5] In the following we thus focus on totally mixed symmetric strategy profiles in which all players vote with probability $x \in (0, 1)$. As shown in Palfrey and Rosenthal (1983, eq. (13)) such a strategy profile is a Nash equilibrium if and only if

$$2c = P(x, s) \tag{1}$$

where $P(x, s)$ is the pivot probability:

$$P(x, s) = \sum_{k=0}^{s-1} \binom{s-1}{k} \binom{s}{k} x^{2k} (1-x)^{2s-2k-1} + \sum_{k=0}^{s-1} \binom{s-1}{k} \binom{s}{k+1} x^{2k+1} (1-x)^{2s-2k-2}. \tag{2}$$

The logic behind the equilibrium conditions (1)–(2) is familiar. For any player i, (2) gives the probability that player i casts the decisive vote when each of the other players votes with probability x: The first sum covers the cases in which k members of the opposing team as well as k other members of player i's own team vote. In this case i's vote resolves a tie and her favored alternative will be selected with probability 1 (rather than 1/2) if she participates. The second sum covers the cases in which $k + 1$ members of the opposing team as well as k other members of i's own team vote. In this case i's vote induces a tie and her favored alternative will be selected with probability 1/2 (rather than 0) if she participates. Because player i gains an expected benefit of 1/2 whenever her vote is pivotal (and a benefit of zero when it is not) the indifference condition for a Nash equilibrium in which i's mixed strategy assigns strictly positive probability to both pure strategies is that her probability of casting the decisive vote is twice the cost of participation. This yields (1) as the necessary and sufficient condition for the strategy profile in which all voters participate with probability $x \in (0, 1)$ to be a symmetric Nash equilibrium.

3. Results

3.1. Preliminaries

We begin by obtaining an alternative representation of the pivot probability $P(x, s)$ as an expectation over conditional pivot probabilities. It is this alternative representation which provides the foundation for all of our subsequent analysis.

Let

$$\phi(k, \ell, s) = \frac{\binom{s-1}{k}\binom{s}{\ell}}{\binom{2s-1}{k+\ell}}, \; k = 0, \ldots, \; s-1, \; \ell = 0, \ldots, s. \tag{3}$$

The expression in (3) is the probability (as determined by the hypergeometric distribution) of k successes in $s - 1$ draws, without replacement, from a population of size $2s - 1$ containing $k + \ell$ successes. Hence, given that the total number of voters among all players but i is $k + \ell$, $\phi(k, \ell, s)$ is the probability that exactly k of these voters are in i's team and exactly ℓ are in the other team. Letting

$$p(j, s) = \begin{cases} \phi(j/2, j/2, s) & \text{if } j = 0, 2, \ldots 2s-2 \\ \phi((j-1)/2, (j+1)/2, s) & \text{if } j = 1, 3, \ldots 2s-1 \end{cases} \tag{4}$$

[5] It is easy to see that for $c > 1/2$ the only Nash equilibrium (either in pure or mixed strategies) is the symmetric one in which no agent participates. In the knife-edge case $c = 1/2$ there is one additional symmetric Nash equilibrium in which all agents participate (and also a multitude of asymmetric equilibria).

then gives the probability that player i is pivotal conditional on there being exactly j voters among all the other players—indeed, if j is even, then player i is pivotal if and only if the same number of other players vote in both teams and if j is odd she is pivotal if and only if there is one less voter among the other players in her team than there are voters in the other team. Throughout the following we will refer to $p(j,s)$ as the conditional pivot probability.

Fix some player i. Given that all other players independently choose to vote with probability x, the probability that there will be exactly j voters among all the other players is $\binom{2s-1}{j}x^j(1-x)^{2s-1-j}$. Multiplying this by the conditional pivot probability $p(j,s)$ and summing over all j gives the pivot probability $P(x,s)$.

Lemma 1. *For all $x \in [0,1]$ and $s \geq 2$, the pivot probability is given by*

$$P(x,s) = \sum_{j=0}^{2s-1} \binom{2s-1}{j} x^j (1-x)^{2s-1-j} p(j,s). \tag{5}$$

The representation of the pivot probability obtained in Lemma 1 is useful because much can be said about the structure of the conditional pivot probabilities $p(j,s)$ and, as we explain in the following subsection, this structure can then be leveraged to infer properties of the pivot probability $P(x,s)$. The following lemma summarizes some key properties that are of relevance for our subsequent analysis. The proof uses well-known properties of the probability mass function of the hypergeometric distribution (cf. Johnson et al., 2005). Alternatively, the results can be verified by straightforward (but tedious) calculations using (3) and (4).

Lemma 2. *For each $s \geq 2$ the conditional pivot probabilities satisfy*[6]

$$p(0,s) = 1 \tag{6}$$

$$p(j,s) = p(2s-1-j,s), \; j = 0,1,\dots,2s-1 \tag{7}$$

$$p(2k,s) = p(2k-1,s), \; k = 1,\dots,s-1 \tag{8}$$

$$p(2k,s) > p(2k+1,s), \; k = 0,\dots,\lceil (s-1)/2 \rceil - 1 \tag{9}$$

Equation (6) is obvious: if there are no voters among the other agents, then the player under consideration is surely pivotal. The equalities in (7) assert that the conditional pivot probabilities are symmetric in the sense that a player is equally likely to be pivotal when there are exactly j voters or exactly j abstainers among the other players. In particular, we also have $p(2s-1,s) = 1$. The meaning of (8) is that increasing an odd number j of voters among the other agents by one leaves the conditional pivot probability unaffected. In contrast, (9) states that increasing an even number of voters among the other players by one strictly decreases the conditional pivot probability if the resulting odd numbers of voters is strictly smaller than the team size, i.e., if $2k+1 < s$. Observe that it follows from the symmetry property in (7) that the inequality in (9) is reversed when $2k+1 > s$ holds.

Fig. 1 illustrates the shape properties of the conditional pivot probabilities $p(j,s)$ established in Lemma 2 and also shows the corresponding pivot probability $P(x,s)$ for the case $s = 8$. The close resemblance between the two functions illustrated in Fig. 1 suggests that shape properties of $P(x,s)$ can be inferred from the shape properties of $p(j,s)$. The following subsection verifies this.

3.2. Existence, number, and comparative statics with respect to participation cost of totally mixed symmetric equilibria

As totally mixed symmetric equilibria are given by the solutions to the equation $2c = P(x,s)$, their existence, number and comparative statics are determined by the shape of the pivot probability $P(x,s)$. Our main result in this subsection (Proposition 1) describes how the pivot probability depends on x for any given team size s. As an immediate corollary, we obtain a precise characterization of the number of totally mixed symmetric equilibria (which can range from zero to two) and a comparative static result describing how the location of the equilibria varies with the participation cost.

The expression for the pivot probability $P(x,s)$ obtained in Lemma 1 is a polynomial in Bernstein form (cf. Farouki, 2012) of degree $2s-1$ with coefficients given by the finite sequence of conditional pivot probabilities $p(j,s)$. To obtain the following proposition we make use of three properties of such polynomials: (1) they preserve symmetry, (2) their derivatives can be again written as polynomials in Bernstein form, and (3) they are variation diminishing.[7] In particular, we apply the symmetry property to infer symmetry of the pivot probability $P(x,s)$ in x from the symmetry of the conditional

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6 In (9) the expression $\lceil (s-1)/2 \rceil$ denotes the ceiling of $(s-1)/2$, that is, the smallest integer greater or equal than $(s-1)/2$. Consequently, $\lceil (s-1)/2 \rceil - 1$ is the largest integer strictly smaller than $(s-1)/2$.

7 Roughly speaking, a transformation is variation diminishing if it reduces the number of sign changes (from positive to negative or vice versa). See Karlin (1968) for precise definitions and Brown et al. (1981) for a gentle introduction in a statistical context. For previous applications in economics, see Jewitt (1987) and Chakraborty (1999).

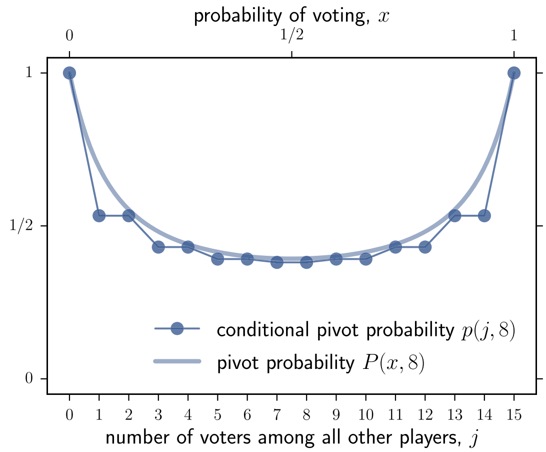

Fig. 1. Conditional pivot probabilities $p(j,s)$ and corresponding pivot probability $P(x,s)$ for $s=8$. As established in Lemma 2, the conditional pivot probabilities $p(j,8)$ are symmetric about $j=(2s-1)/2$ and u-shaped. Proposition 1 establishes that the corresponding pivot probability $P(x,8)$ inherits these shape properties, so that it is both symmetric about $x=1/2$ and u-shaped.

pivot probabilities established in Lemma 2. Using the derivative property we can express the derivative of $P(x,s)$ with respect to x as a polynomial in Bernstein form with coefficients given by a positive multiple of the difference between adjacent conditional pivot probabilities. Due to (8) and (9) in the statement of Lemma 2 we know that this finite sequence of coefficients has exactly one sign change (from negative to positive). An application of the variation diminishing property then implies that the derivative of $P(x,s)$ has exactly one root. By symmetry, this root is located at $x=1/2$. We thus obtain:

Proposition 1. *For each $s\geq 2$, the pivot probability $P(x,s)$ satisfies $P(0,s)=P(1,s)=1$ and is (1) symmetric in x: $P(x,s)=P(1-x,s)$ for all $x\in[0,1]$ and (2) strictly decreasing in x in the interval $[0,1/2]$ and strictly increasing in x in the interval $[1/2,1]$.*

For $x\in[0,1]$ let

$$c(x,s)=P(x,s)/2, \tag{10}$$

so that the equilibrium condition (1) can be restated as $c=c(x,s)$. For $x\in(0,1)$ we can then think of $c(x,s)$ as the (uniquely determined) participation cost for which x is a totally mixed symmetric equilibrium. From Proposition 1 it is clear that for given s the function $c(x,s)$ has a unique minimum at $x=1/2$. We denote this minimum by

$$\underline{c}(s)=c(1/2,s)=\left(\frac{1}{2}\right)^{2s-1}\binom{2s-1}{s}<\frac{1}{2}, \tag{11}$$

where the second equality is from Proposition 6(a) in Palfrey and Rosenthal (1983) and the final inequality holds because $x=1/2$ is the unique minimizer of $c(x,s)$, which satisfies $c(0,s)=c(1,s)=1/2$. It is then immediate that for $c<\underline{c}(s)$ no totally mixed symmetric equilibrium exists and that for $c=\underline{c}(s)$ the unique such equilibrium is $x=1/2$. On the other hand, continuity of $c(x,s)$ in x together with the monotonicity properties established in Proposition 1 ensures that for $c\in(\underline{c}(s),1/2)$ there are exactly two solutions to the equilibrium condition $c=c(x,s)$, with of one these solutions being smaller than 1/2 and the other being larger than 1/2. Further, from the symmetry property established in Proposition 1 these two solutions have the same distance from 1/2 and, due to the u-shape of $P(x,s)$, this distance is monotonically increasing from 0 to 1/2 as c increases from $\underline{c}(s)$ to 1/2. We summarize these findings as follows (see Fig. 2 for illustration):

Corollary 1. *For each $s\geq 2$ the following holds:*

1. *If $c<\underline{c}(s)$, there exists no totally mixed symmetric Nash equilibrium.*
2. *If $c=\underline{c}(s)$, $x=1/2$ is the unique totally mixed symmetric Nash equilibrium.*
3. *If $c\in(\underline{c}(s),1/2)$, there exist exactly two totally mixed symmetric Nash equilibria $x_{\mathrm{L}}(c,s)$ and $x_{\mathrm{H}}(c,s)$, satisfying $0<x_{\mathrm{L}}(c,s)<1/2<x_{\mathrm{H}}(c,s)<1$. Further, $x_{\mathrm{L}}(c,s)+x_{\mathrm{H}}(c,s)=1$ holds for all c in the indicated interval, $x_{\mathrm{L}}(c,s)$ is strictly decreasing in c, and $x_{\mathrm{H}}(c,s)$ is strictly increasing in c with $\lim_{c\to\underline{c}(s)}x_{\mathrm{L}}(c,s)=\lim_{c\to\underline{c}(s)}x_{\mathrm{H}}(c,s)=1/2$, $\lim_{c\to 1/2}x_{\mathrm{L}}(c,s)=0$, and $\lim_{c\to 1/2}x_{\mathrm{H}}(c,s)=1$.*

Because $\underline{c}(s)<1/2$ holds, the range of cost values such that both the low turnout equilibrium $x_{\mathrm{L}}(c,s)$ and the high turnout equilibrium $x_{\mathrm{H}}(c,s)$ exist is non-empty for all team sizes $s\geq 2$. Further, as $\lim_{s\to\infty}\underline{c}(s)=0$ holds (cf. Palfrey and Rosenthal, 1983, Proposition 6(c)), it follows from Corollary 1.3 that for all $c\in(0,1/2)$ there exists a critical team size S

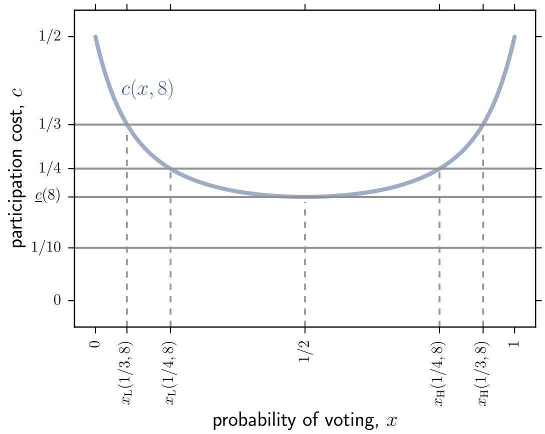

Fig. 2. Illustration of Corollary 1 for team size $s = 8$ and participation costs $c = \{1/10, \underline{c}(8), 1/4, 1/3\}$. For $c < \underline{c}(8)$ the equation $c = c(x, 8)$ has no solution, so that no totally mixed symmetric Nash equilibrium exists. This is the case, for instance, if $c = 1/10$. For $c = \underline{c}(8)$, the unique such equilibrium is $x = 1/2$. For $c \in (\underline{c}(8), 1/2)$ both a low turnout equilibrium $x_L(c, 8)$ and high turnout equilibrium $x_H(c, 8)$ exist. In this case, the probability of voting is strictly decreasing in c at the low turnout equilibrium and strictly increasing in c at the high turnout equilibrium. For instance, both $x_L(1/3, 8) < x_L(1/4, 8)$ and $x_H(1/3, 8) > x_H(1/4, 8)$ hold.

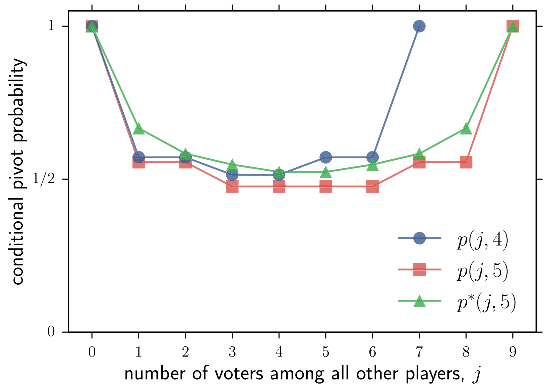

Fig. 3. Conditional pivot probabilities $p(j, s)$, $p(j, s+1)$, and degree-elevated conditional pivot probabilities $p^*(j, s+1)$ for $s = 4$. As established in Lemma 3, $p(j, 4) > p(j, 5)$ holds for all $j = 1, \ldots, 7$. Additionally, it can be shown from Lemmas 2 and 3 that $p^*(j, 5) > p(j, 5)$ holds for all $j = 1, \ldots, 2s$ (see the proof of Proposition 2 in Appendix A).

such that for all $s \geq S$ both the low turnout equilibrium $x_L(c, s)$ and the high turnout equilibrium $x_H(c, s)$ (and no other totally mixed symmetric Nash equilibria) exist. In the following subsection we investigate how, provided that they exist, a change in team size affects the location of these two equilibria.

3.3. Team-size effect

We proceed as in the previous subsection by first exploiting Lemmas 1 and 2 to determine how the pivot probability $P(x, s)$ depends on team size s (Propositions 2 and 3) and then exploiting these results to determine the team-size effect, that is, how the location of totally mixed symmetric equilibria depends on the size of the teams.

The results we obtain for the pivot probability are very intuitive: given any symmetric strategy profile in which all players mix with probability $x \in (0, 1)$, a larger team size results in a lower pivot probability and this pivot probability converges to zero as team size goes to infinity. Establishing these results, however, is a non-trivial task and requires a number of steps. We begin with the observation that conditional pivot probabilities satisfy a monotonicity property.

Lemma 3. *For each $s \geq 2$ the conditional pivot probabilities satisfy*

$$p(j, s) > p(j, s+1), \; j = 1, \ldots, 2s-1 \tag{12}$$

$$p(2s-1-j, s) > p(2s+1-j, s+1), \; j = 1, \ldots, 2s-1 \tag{13}$$

We can offer a partial intuition for the result in Lemma 3 (see also Fig. 3 for an illustration). First, observe that by the symmetry of the conditional pivot probabilities (equation (7) in Lemma 2) the two statements in (12) and (13) are actually equivalent. (We state both sets of inequalities as both will be used in subsequent steps of the argument.) Hence, we may focus on the inequalities in (12). Consider the extreme case $j = 2s - 1$. In this case the conditional pivot probability is 1 when team size is s because all other players vote, whereas it is strictly below one when team size is $s + 1$ as there is the possibility that the two abstainers among the other agents happen to be in the same team. This implies $p(2s - 1, s) > p(2s - 1, s + 1)$. At the other extreme, there is only one voter among the other players ($j = 1$). Because there is one less other player in the own team of any player i than in the opposing team, the probability that the voter is in the opposing team, which is the case in which player i is pivotal, exceeds one-half for any team size s. Indeed, the relevant probability is simply given by $p(1, s) = s/(2s - 1)$, i.e., the probability that a randomly sampled co-player is in the other team rather than in player i's team. As $s/(2s - 1)$ is strictly decreasing in s, this yields $p(1, s) > p(1, s + 1)$. Considering these extreme cases identifies two distinct effects of an increase in team size on the conditional pivot probabilities. For j between these extremes one would expect both of these effects to be at work and hence, as both effects tend to lower the conditional pivot probabilities when team size increases, the result in (12).

The monotonicity of the conditional pivot probabilities established in Lemma 3 is suggestive but does not immediately imply a corresponding monotonicity property for the pivot probability $P(x, s)$. The difficulty is that for given x the binomial distribution governing j depends, of course, on team size. To establish the following proposition, we thus exploit a further result for polynomials in Bernstein form, namely the degree elevation formula (Farouki, 2012, p. 391). This allows us to write $P(x, s)$ as a polynomial of degree $2s+1$ featuring the same binomial probabilities as those appearing in $P(x, s+1)$ and coefficients $p^*(j, s + 1)$ that are convex combinations of the conditional pivot probabilities $p(j, s)$. Lemmas 2 and 3 imply that for $j = 1, \ldots, 2s$ these transformed coefficients are strictly greater than the conditional pivot probabilities $p(j, s + 1)$ (see Fig. 3 for an illustration). Since decreasing the coefficients of a polynomial in Bernstein form with given degree clearly decreases the polynomial, we thus obtain:

Proposition 2. *For every $x \in (0, 1)$ the pivot probability $P(x, s)$ is strictly decreasing in team size s.*

Having established that pivot probabilities are strictly decreasing in team size for any totally mixed symmetric strategy profile, we turn to establish that these pivot probabilities converge to zero. The argument here is fairly standard. It proceeds by (i) showing that the probability that the ratio $j/(2s - 1)$ lies outside a δ-neighborhood of x converges to zero as team size goes to infinity and (ii) providing an upper bound for the conditional pivot probabilities when the ratio $j/(2s - 1)$ lies in the δ-neighborhood that also converges to zero as team size goes to infinity.

Proposition 3. *For every $x \in (0, 1)$ the pivot probability $P(x, s)$ satisfies $\lim_{s\to\infty} P(x, s) = 0$.*

Consider now any $c \in (0, 1/2)$ and recall the definitions of the critical cost levels $c(x, s)$ and $\underline{c}(s)$ from equations (10) and (11). From Propositions 2 and 3 we know that $\underline{c}(s)$ is strictly decreasing in s with $\lim_{s\to\infty} \underline{c}(s) = 0$.[8] It is then an immediate implication of Corollary 1 that there exists a critical team size S such that two totally mixed symmetric equilibria exist if and only if team size is at least S. Further, provided that $s \geq S$ holds, the low turnout equilibrium $x_L(c, s)$ is given by the unique solution of the equation $c = c(x, s)$ in the interval $(0, 1/2)$ and the high turnout equilibrium is given by the unique solution of the same equation in the interval $(1/2, 1)$. Combining the monotonicity properties of the pivot probabilities $P(x, s)$ established in Propositions 1 and 2, an increase in team size then causes both the low turnout and the high turnout equilibrium to move away from $1/2$, so that $x_L(c, s)$ is a strictly decreasing and $x_H(c, s)$ is a strictly increasing function of team size. It then follows from Proposition 3 that $x_L(c, s)$ converges to 0 and $x_H(c, s)$ converges to 1 as team size converges to infinity. We have thus established (see Fig. 4 for an illustration):

Corollary 2. *For every $c \in (0, 1/2)$ there exists a team size S such that $c > \underline{c}(s)$ holds if and only if $s \geq S$, so that the two totally mixed symmetric Nash equilibria $x_L(c, s)$ and $x_H(c, s)$ described in Corollary 1.3 exist if and only if $s \geq S$. Further, these equilibria satisfy $x_L(c, s') < x_L(c, s) < x_H(c, s) < x_H(c, s')$ for all team sizes $s' > s \geq S$, $\lim_{s\to\infty} x_L(c, s) = 0$, and $\lim_{s\to\infty} x_H(c, s) = 1$.*

At first sight the result that for given participation cost $c \in (0, 1/2)$ the low turnout and the high turnout equilibrium exist if and only if team size is *above* a critical threshold may seem puzzling. The explanation is, however, straightforward: Non-existence of these equilibria only arises if the pivot probability $P(x, s)$ is strictly greater than twice the participation cost for all x, indicating that every player finds it worthwhile to participate no matter what the participation probability of the other players is. In other words: existence of a totally mixed symmetric equilibrium requires that the pivot probability $P(x, s)$ is sufficiently *low* for some x as otherwise there is no chance to equalize the expected benefit and the cost from voting. As one would expect (and we have formally demonstrated in Proposition 2) pivot probabilities are strictly decreasing in team size, so that the indifference condition (1) has a solution if and only if team size is large enough.

[8] As noted after the statement of Corollary 1, the result $\lim_{s\to\infty} \underline{c}(s) = 0$ has already been established in Palfrey and Rosenthal (1983).

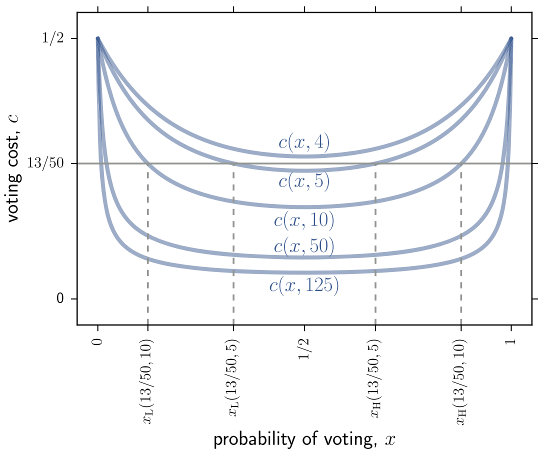

Fig. 4. Illustration of Corollary 2 for participation cost $c = 13/50$. In this case the critical team size is $S = 5$. For all team sizes $s \geq S$, the probability of voting decreases with team size at the low turnout equilibrium $x_L(c, s)$ and increases with team size at the high turnout equilibrium $x_H(c, s)$. For instance, $x_L(13/50, 10) < x_L(13/50, 5)$ and $x_H(13/50, 10) > x_H(13/50, 5)$.

4. Discussion

We have characterized the symmetric Nash equilibria of the symmetric voter participation game with complete information introduced by Palfrey and Rosenthal (1983), confirming their conjecture about the existence, multiplicity, and comparative statics properties of totally mixed symmetric equilibria and providing additional comparative statics results.

As mentioned in Palfrey and Rosenthal (1983, p. 33), the truth of their conjecture implies further results for the case in which the participation cost is lower than the critical value required for the existence of a totally mixed symmetric equilibrium. In particular, for this case our results imply the existence of exactly two equilibria in which all members of team T_1 vote with probability $0 < x_1 < 1$ and all members of team T_2 vote with probability $x_2 = 1 - x_1$. Similarly, it is clear that any totally mixed equilibrium for the symmetric case that we have studied here also induces an equilibrium in a game in which the two teams supporting the two different alternatives have different sizes: if these team sizes are, say, s_1 and s_2 with $s_2 > s_1$, then a profile in which (i) $s_2 - s_1$ voters of the second team abstain and (ii) all other voters play one of the mixed strategy equilibria for team size s_1 yields an equilibrium for the overall game.

The key insight underlying our analysis is that for any symmetric strategy profile the pivot probability (which determines a voter's incentive to participate) can be represented as a polynomial in Bernstein form with coefficients given by conditional pivot probabilities. While the results we obtain here hinge on the specific properties of the conditional pivot probabilities, the theory of polynomials in Bernstein form is, as we have already argued in the Introduction, a promising tool to further the analysis of symmetric participation games.

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Appendix A

Proof of Lemma 1. Setting $j = 2k$ and using (4), we have

$$\sum_{k=0}^{s-1} \binom{s-1}{k}\binom{s}{k} x^{2k}(1-x)^{2s-2k-1} = \sum_{j=0,2,\ldots,2s-2} \binom{s-1}{j/2}\binom{s}{j/2} x^j (1-x)^{2s-1-j}$$
$$= \sum_{j=0,2,\ldots,2s-2} \binom{2s-1}{j} x^j (1-x)^{2s-1-j} p(j,s)$$

Similarly, setting $j = 2k + 1$ and using (4), we have

$$\sum_{k=0}^{s-1} \binom{s-1}{k}\binom{s}{k+1} x^{2k+1}(1-x)^{2s-2k-2} = \sum_{j=1,3,\ldots 2s-1} \binom{s-1}{(j-1)/2}\binom{s}{(j+1)/2} x^j (1-x)^{2s-1-j}$$
$$= \sum_{j=1,3,\ldots,2s-1} \binom{2s-1}{j} x^j (1-x)^{2s-1-j} p(j,s).$$

Substituting the above identities into the definition of $P(x, s)$ in (2) yields (5). □

Proof of Lemma 2. From (4) we have $p(0,s)=\phi(0,0,s)$ and from (3) we have $\phi(0,0,s)=1$, so that $p(0,s)=1$ holds for all $s\geq 2$, thus establishing identity (6).

For the remainder of the proof we refer to properties of the probability mass function of the hypergeometric distribution listed in Johnson et al. (2005, pp. 265–266). Consequently, it is useful to observe that in their notation for the probability mass function of the hypergeometric distribution the definition of ϕ in (3) becomes

$$\phi(k,\ell,s)=f(k\mid k+\ell,s-1,s)$$

and the definition of p in (4) becomes

$$p(j,s)=\begin{cases} f(j/2\mid j,s-1,s) & \text{if } j=0,2,\dots 2s-2\\ f((j-1)/2\mid j,s-1,s) & \text{if } j=1,3,\dots 2s-1.\end{cases} \tag{14}$$

Suppose j is even. Then $2s-1-j$ is odd, so that using (14) the equality in (7) becomes

$$f(j/2\mid j,s-1,s)=f(s-1-j/2\mid 2s-1-j,s-1,s),$$

which is immediate from Johnson et al. (2005, eq. (6.56), second line). The argument when j is odd is analogous.

Using (14) the equality in (8) becomes

$$f(k\mid 2k,s-1,s)=f(k-1\mid 2k-1,s-1,s).$$

To establish this, it suffices to note that from Johnson et al. (2005, eq. (6.54)) we have

$$f(k-1\mid 2k,s-1,s)=\frac{k}{k+1}f(k-1\mid 2k-1,s-1,s)$$

and from Johnson et al. (2005, eq. (6.52)) we have

$$f(k\mid 2k,s-1,s)=\frac{k+1}{k}f(k-1\mid 2k,s-1,s).$$

Finally, using (14) the inequality appearing in (9) becomes

$$f(k\mid 2k,s-1,s)>f(k\mid 2k+1,s-1,s).$$

From Johnson et al. (2005, eq. (6.54)) this inequality is equivalent to

$$\frac{(s-k)(2k+1)}{(k+1)(2s-1-2k)}<1\Leftrightarrow k<\frac{s-1}{2},$$

which is the desired result. □

Proof of Proposition 1. From (5) in Lemma 1 it is immediate that $P(0,s)=p(0,s)$ and $P(1,s)=p(2s-1,s)$ holds, so that $P(0,s)=P(1,s)=1$ follows from Lemma 2.

By the symmetry of the Bernstein basis polynomials (Farouki, 2012, p. 389) and the symmetry of the conditional pivot probabilities, that is (7) in Lemma 2, we have

$$\binom{2s-1}{j}x^j(1-x)^{2s-1-j}p(j,s)=\binom{2s-1}{2s-1-j}(1-x)^{2s-1-j}x^j p(2s-1-j,s)$$

for all $x\in[0,1]$ and $j=0,\dots,2s-1$ and thus

$$\sum_{j=0}^{2s-1}\binom{2s-1}{j}x^j(1-x)^{2s-1-j}p(j,s)=\sum_{j=0}^{2s-1}\binom{2s-1}{j}(1-x)^j x^{2s-1-j}p(j,s),$$

for all $x\in[0,1]$. From equation (5) in Lemma 1 this implies the symmetry property $P(x,s)=P(1-x,s)$ for all $x\in[0,1]$.

Let $P'(x,s)$ denote the derivative of $P(x,s)$ with respect to x. From the derivative property of polynomials in Bernstein form (Farouki, 2012, p. 391) we have

$$P'(x,s)=\sum_{j=0}^{2s-2}\binom{2s-2}{j}x^j(1-x)^{2s-2-j}\left[(2s-1)\Delta p(j,s)\right], \tag{15}$$

where, for $j=0,1\dots,2s-2$,

$$\Delta p(j,s)=p(j+1,s)-p(j,s).$$

From (8) in Lemma 2 the equality $\Delta p(j,s)=0$ holds whenever j is odd. From (9) we have $\Delta p(j,s)<0$ whenever j is even and $j<s-1$ holds, whereas (by symmetry) $\Delta p(j,s)>0$ holds whenever j is even and $j>(s-1)$ holds. If $j=s-1$ is even, then $\Delta p(j,s) = 0$ is implied by the symmetry of the conditional pivot probabilities. It follows that the finite sequence of the coefficients $\Delta p(j,s)$ has exactly one sign change (when zero coefficients are ignored). Consequently, the variation diminishing property of polynomials in Bernstein form (Farouki, 2012, p. 390) implies that the equation $P'(x,s) = 0$ has exactly one solution in the interval $(0,1)$. By the symmetry of $P(x,s)$ in x this solution must occur at $x = 1/2$. Finally, as (15) implies $P'(0,s) = (2s-1)\Delta p(0,s) < 0$, we obtain $P'(x,s) < 0$ for all $x \in [0,1/2)$ and (by symmetry) $P'(x,s) > 0$ for all $x \in (1/2,1]$. □

Proof of Lemma 3. From (7) it suffices to prove (12). Further, from (8) it suffices to establish the inequalities in (12) for odd j. Setting $j = 2k+1$ and using (3) and (4) we thus have to show

$$\frac{\binom{s-1}{k}\binom{s}{k+1}}{\binom{2s-1}{2k+1}} > \frac{\binom{s}{k}\binom{s+1}{k+1}}{\binom{2s+1}{2k+1}}$$
$$\Leftrightarrow \binom{s-1}{k}\binom{s}{k+1}\binom{2s+1}{2k+1} > \binom{s}{k}\binom{s+1}{k+1}\binom{2s-1}{2k+1}$$

for $k = 0, 1, \ldots, s-1$. Using the factorial formula for the binomial coefficients and eliminating identical factorials this is

$$\frac{(s-1)!(2s+1)!}{(s-1-k)!(s-1-k)!(2s-2k)!} > \frac{(s+1)!(2s-1)!}{(s-k)!(s-k)!(2s-2k-2)!}$$
$$\Leftrightarrow (2s+1)2s(s-k)(s-k) > (s+1)s(2s-2k)(2s-2k-1)$$
$$\Leftrightarrow (2s+1)(s-k) > (s+1)(2s-2k-1)$$
$$\Leftrightarrow k+1 > 0,$$

establishing the desired result. □

Proof of Proposition 2. For convenience, define $p(-2,s) = p(-1,s) = 1$ and $p(2s+1,s) = p(2s,s) = 1$, so that $p(-2,s) = p(-1,s) = p(0,s)$ and $p(2s+1,s) = p(2s,s) = p(2s-1,s)$ hold. Applying the degree elevation formula for polynomials in Bernstein form (Farouki, 2012, p. 391) twice we can write

$$P(x,s) = \sum_{j=0}^{2s+1} \binom{2s+1}{j} x^j (1-x)^{2s+1-j} p^*(j,s+1), \tag{16}$$

where the elevated coefficients $p^*(j,s+1)$ are determined from the formula in the last line of Farouki (2012, p. 391). In particular, for all $j = 0, 1, \ldots, s$ there exist $\lambda_{-2} > 0$, $\lambda_{-1} > 0$, and $\lambda_0 > 0$ such that $\lambda_{-2} + \lambda_{-1} + \lambda_0 = 1$ and

$$p^*(j,s+1) = \lambda_{-2} p(j-2,s) + \lambda_{-1} p(j-1,s) + \lambda_0 p(j,s) \tag{17}$$

hold. Similarly, for all $j = 0, 1, \ldots s$ there exist $\mu_0 > 0$, $\mu_1 > 0$, and $\mu_2 > 0$ such that $\mu_0 + \mu_1 + \mu_2 = 1$ and

$$p^*(2s+1-j,s+1) = \mu_0 p(2s-1-j,s) + \mu_1 p(2s-j,s) + \mu_2 p(2s+1-j,s) \tag{18}$$

hold. Lemma 2 implies that for all $j = 0, 1, \ldots, s$ we have

$$p(j-2,s) \geq p(j-1,s) \geq p(j,s),$$

and

$$p(2s-1+j,s) \geq p(2s-j,s) \geq p(2s-1-j,s).$$

Applying these inequalities on the right sides of (17) and (18) we obtain

$$p^*(j,s+1) \geq p(j,s) \text{ and } p^*(2s+1-j,s+1) \geq p(2s-1-j,s)$$

for $j = 0, 1, \ldots, s$. Lemma 3 then implies

$$p^*(j,s+1) > p(j,s+1) \text{ and } p^*(2s+1-j,s+1) > p(2s+1-j,s+1) \tag{19}$$

for $j = 1, \ldots, s$. As we also have $p^*(0,s+1) = p(0,s+1) = 1$ and $p^*(2s+1,s+1) = p(2s+1,s+1) = 1$, (19) implies

$$\sum_{j=0}^{2s+1} \binom{2s+1}{j} x^j (1-x)^{2s+1-j} p^*(j,s+1) > \sum_{j=0}^{2s+1} \binom{2s+1}{j} x^j (1-x)^{2s+1-j} p(j,s+1)$$

for all $x \in (0,1)$. Using (16) and Lemma 1 this is equivalent to $P(x,s) > P(x,s+1)$ for all $x \in (0,1)$. □

Proof of Proposition 3. Given any $x \in (0,1)$, there exist $\delta > 0$ and $\epsilon > 0$ such that $\epsilon < x - \delta < x + \delta < 1 - \epsilon$ holds. Fix such a δ and ϵ. From equation (5) in Lemma 1 we can then write

$$P(x,s) = \sum_{\{j:|\frac{j}{2s-1}-x|\geq\delta\}} \binom{2s-1}{j} x^j (1-x)^{2s-1-j} p(j,s) + \sum_{\{j:|\frac{j}{2s-1}-x|<\delta\}} \binom{2s-1}{j} x^j (1-x)^{2s-1-j} p(j,s).$$

As $p(j,s) \leq 1$ holds for all j and s, a standard bound for the binomial probability distribution (e.g. Chang and Sederberg, 1997, Theorem 25.5) implies that the first sum in the above expression is smaller than $1/(4(2s-1)\delta^2)$ for all $s \geq 2$. Letting k_s denote the largest integer k satisfying the inequality $2k/(2s-1) \leq \epsilon$, we further have from Lemma 2 that the second sum in the above expression is bounded above by $p(2k_s, s)$. Therefore, to prove the proposition it suffices to show that $p(2k_s, s)$ converges to zero as s converges to infinity.

It is clear that for $s \to \infty$ we also have $k_s \to \infty$ with $\lim_{s\to\infty} k_s/s = \epsilon \in (0,1)$. We may write

$$\begin{aligned} p(2k_s, s) &= \frac{\binom{s-1}{k_s}\binom{s}{k_s}}{\binom{2s-1}{2k_s}} \\ &= \binom{2k_s}{k_s} \frac{\prod_{i=1}^{k_s}(s-i)\prod_{i=1}^{k_s}(s+1-i)}{\prod_{i=1}^{2k_s}(2s-i)} \\ &= \binom{2k_s}{k_s} \frac{s(s-k_s)}{(2s+1-2k_s)(2s-2k_s)} \prod_{i=1}^{k_s-1} \frac{(s-i)^2}{(2s+1-2i)(2s-2i)}. \end{aligned}$$

Observing that

$$\frac{s(s-k_s)}{(2s+1-2k_s)(2s-2k_s)} \prod_{i=1}^{k_s-1} \frac{(s-i)^2}{(2s+1-2i)(2s-2i)} \leq \left(\frac{1}{2}\right)^{2k_s} \frac{s}{s-k_s},$$

where the inequality holds because the first term on the left side is smaller than $(1/4)(s/(s-k_s))$ and every term in the following product is smaller than $1/4$, we thus obtain

$$p(2k_s, s) \leq \binom{2k_s}{k_s} \left(\frac{1}{2}\right)^{2k_s} \frac{s}{s-k_s}.$$

Because $s/(s-k_s)$ converges to the finite limit $1/(1-\epsilon)$, it then suffices to establish

$$\lim_{k_s\to\infty} \binom{2k_s}{k_s} \left(\frac{1}{2}\right)^{2k_s} = 0$$

to obtain the desired result. But this is immediate from Stirling's approximation for binomial coefficients of the form $\binom{2n}{n}$ for large n. □

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